

Well posedness and stationary solutions of a neural field equation with synaptic plasticity

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Abstract

We consider the initial value problem associated to the neural field equation of Amari type with plasticity

$$u_t(x, t) = -u(x, t) + \int_{\Omega} w(x, y)[1 + \gamma g(u(x, t) - u(y, t))]f(u(y, t)) dy, \quad (x, t) \in \Omega \times (0, \infty),$$

where $\Omega \subset \mathbb{R}^m$, f and g are bounded and continuously differentiable functions with bounded derivative, and $\gamma \geq 0$ is the plasticity synaptic coefficient. We show that the problem is well posed in $C_b(\mathbb{R}^m)$ and $L^1(\Omega)$ with Ω compact. The proof follows from a classical fixed point argument when we consider the equation's flow. Strong convergence of solutions in the no plasticity limit ($\gamma \rightarrow 0$) to solutions of Amari's equation is analysed. Finally, we prove existence of stationary solutions in a general way. As a particular case, we show that the Amari's model, after learning, leads to the stationary Schrödinger equation for a type of gain modulation.

1 Introduction

Neural fields equations are models of the activity of spatially structured populations of neurons in the continuum limit [1]. In these models, the population of neurons is represented by a spatial domain $\Omega \subset \mathbb{R}^m$ and the activity at position $x \in \Omega$ corresponds to a time varying scalar field $u(x, t)$. The dynamics of the population is described, in general, by an integrodifferential equation. Multiple interacting populations are thus described by coupled systems of equations. The ultimate form of the equation that describes the dynamics of the field depends

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on the kind of phenomena being modeled, which range from single neuron properties (e.g. refractory periods [2], membrane dynamics [1], adaptation[3]) to synaptic or population features (e.g. transmission delays [4], plasticity[5, 6, 7], inhibitory/excitatory connectivity[2])

Neural fields have been used to model EEG rhythms, epilepsy, binocular rivalry and a broad range of cognitive phenomena[1, 8]. A recent account of the research and challenges in the area can be found in [1]. Two commonly used neural fields models are the Wilson and Cowan model [2] and the Amari neural field equation [9]. In the Wilson and Cowan model, the field variable corresponds to the proportion of firing neurons. This model includes refractory periods and nonlocal interactions which are usually removed in modern mean field theories. The Amari model can be voltage based (i.e. the field variable represents the membrane potential) or rate based (i.e. the field variable represents the firing rate); this model can be seen as a special case of the Wilson and Cowan one in which the refractory period is removed but nonlocal interactions are preserved.

In this work we focus in a voltage based neural field model of Amari type with plasticity. In the traditional voltage based single population Amari model[9], the dynamics of the membrane voltage $u(x, t)$ is described by the following nonlinear integrodifferential equation of Hammerstein type

$$u_t(x, t) + u(x, t) = \int_{\mathbb{R}^m} \omega(x, y) f(u(y, t)) dy. \quad (1)$$

The left hand part of the equation represents a first order membrane dynamics (exponential decay of incoming pulses). The right hand side represents the synaptic inputs from other neurons in the population. The kernel of the integral $w(x, y)$ is interpreted as the weight or strength between the *pre-synaptic* neuron at position y and the *post-synaptic* one at position x , and it is called *synaptic kernel*. The sign of the kernel determines whether the synapse is excitatory or inhibitory. Commonly used kernels are, the purely excitatory $w(\delta) = \exp(-|\delta|)/2$ and the so called mexican hat $w(\delta) = (1 - |\delta|) \exp(-|\delta|)$, which represents long range inhibitory - short range excitatory connections[10, 1]; it is usually assumed $w(x, y) = w(|x - y|)$ (i.e. isotropic connectivity). $f(u(x, t))$ stands for the firing rate of the neuron as a function of the membrane voltage; it can be understood as the probability of neuron x firing an action potential at time t . Commonly used firing rate functions are sigmoidal functions like $f(\delta) = 1/(1 + \exp(-\delta))$ or $f(\delta) = \arctan(\delta)$, or linear/piece-wise linear functions[8, 10].

Plasticity is introduced here by modifying the kernel in an Hebbian, activity dependent manner (i.e. neurons that fire together, increase their connection strength)[11, 6, 7]; that is, the term plasticity is understood in a broad sense that include many synaptic mechanisms (e.g. facilitation, potentiation, depression,

etc.). In this paper we study the particular dynamic kernel model of Abbasian et al. [6]

$$u_t(x, t) + u(x, t) = \int_{-\infty}^{\infty} \omega_{g, \gamma}(x, y) f(u(y, t)) dy, \quad (2)$$

with

$$\omega_{g, \gamma}(x, y) = \omega(x, y)[1 + \gamma g(u(x, t) - u(y, t))], \quad (3)$$

where g , called the *plasticity or learning kernel*, is the gaussian function $g(\delta) = e^{-\delta^2}$ and $\gamma \geq 0$ which we will refer to as the *synaptic plasticity coefficient*. It is important to note that, in this model, $u(x, t)$ is the membrane potential as previously stated and, therefore, the voltage and not the firing rate is used to modify the synaptic strength.

The parameter γ represents the increase in the synaptic strength when the pre and post-synaptic neurons fire together at the same rate ($g(u(x, t) - u(y, t)) \approx g(0) = 1$). This coefficient is related to the learning rate in discrete neural models[11]. When the membrane potentials $u(x, t)$ and $u(y, t)$ are similar, the connection of the neurons will be strengthened; when different, the activity is assumed to be uncorrelated and the behavior approaches that of the Amari's model ($g(\delta) \rightarrow 0$ when $\delta \rightarrow \infty$). Finally, when both neurons are silent, the functional form of the firing rate function have to be chosen in order avoid any unexpected behavior due to false increases in the synaptic strength.

Due to the particular form of the plasticity kernel, after the learning has occurred (i.e. in the stationary case, $t \rightarrow \infty$), it can be divided in a pre-synaptic and post-synaptic contribution $g(u_{\infty}(x) - u_{\infty}(y)) = \phi_{\text{post}}(x)\phi_{\text{pre}}(y)$; as seen in section 5.1. The pre-synaptic contribution defines a *multiplicative gain field*, that is, a multiplicative modulation in the slope of the firing rate function in a position dependent way. Gain modulation is an important phenomenon related with neural computations [12, 13]. It has been proposed to play a major role in attention, propagation of epileptic seizures and the stability of the underlying networks[14, 15]. Gain modulation emerges here as a by-product of learning, therefore, the function $\phi_{\text{pre}}(y)$ represents the increase in the neurotransmitter release probability and $\phi_{\text{post}}(x)$ represents the increase in receptor density in the post-synaptic neuron (among other membrane and internal changes in the cell); both observed products of learning in biological neurons [5]

To study the effects of the learned patterns on the dynamics of neural fields, the learning is deactivated but the stationary kernel is preserved and decomposed as previously described. The new kernel can be used in a second neural field equation in which the learned structured does not change any more but is fixed to the stationary solution of (2)

$$v_t(x, t) + v(x, t) = \frac{1}{2\lambda} \int_{\Omega} w(x, y) G(u_{\infty}(x) - u_{\infty}(y)) f(v(y, t)) dy. \quad (4)$$

We introduce a particular form of this equation in connection to the plasticity model being studied

$$v_t(x, t) + v(x, t) = \int_{\Omega} e^{-\lambda|x-y|} \phi_{\text{pre}}(y) v(y, t) dy \quad (5)$$

with $\omega(x) = \frac{1}{\lambda} e^{-\lambda|x|}$ and $\lambda > 0$. In section 5.1 we study some of its properties and those results are extended in an upcoming paper [23].

Most of the theoretical research on neural fields have been focused in studying the properties of particular solutions, like resting state, bumps or traveling waves [3, 1, 8]. The work of Abbassian et al. is also oriented in that direction [6]. To our knowledge, the first rigorous approach to the existence of general solutions of the Amari model (1) was done by Potthast and Graben [16]. Similar and further results can be found in [17, 18]. Further analysis, like asymptotic behavior in bounded and unbounded domains can be found in [19, 20]. For a smooth firing rate f , a well defined synaptic kernel ω and initial datum $u(x, 0) = u_0(x)$, $u_0 \in C_b(\mathbb{R}^n)$ (the space of continuous bounded functions), in [16] they showed that there exist a function $v \in C([0, +\infty) : C_b(\mathbb{R}^n))$, solution of (1) in $\mathbb{R}^n \times (0, +\infty)$ and $v(x, 0) = u_0(x)$. This is a global result for the Cauchy problem associated to (1). If f is discontinuous (e.g. a Heaviside function), they showed that this method is not valid. Then, with another kind of restrictions on the kernel and more specialized spaces, they obtained weaker solutions using a compactness argument.

In this work we study the Initial Value Problem (IVP) associated with (2)

$$u_t(x, t) + u(x, t) = \int_{\Omega} \omega_{g,\gamma}(x, y) f(u(y, t)) dy, \quad t > 0, x \in \Omega, \quad (6)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega \subset \mathbb{R}^m, \quad (7)$$

with $m \geq 1$. We follow the ideas of Potthast and Graben [16] to show that the IVP (6)-(7) is globally well posed in $C_b(\mathbb{R}^n)$ and $L^1(\Omega)$ for Ω bounded. We also show that the solutions for the Amari model (1) are a good approximation for the model with plasticity of Abbasian et al. (an assumption used for those authors in proving their results). More precisely, we show that solutions of (6) converge uniformly to solutions of (1) when $\gamma \rightarrow 0$. In the last section we improve the results of [6] by showing the existence of stationary solutions of (6) for small γ and $\Omega \subset \mathbb{R}^m$ compact. Finally, the effect of learned gain fields is studied by introducing the Gain Field Equation (133) and we show that, for certain gain fields, it corresponds to the stationary Schrödinger equation.

2 Preliminary

We begin this section defining the concept of well posedness of a Initial Value Problem (IVP).

Consider the following Cauchy problem:

$$\begin{cases} u_t(t) &= F(t, u(t)) \in X, \\ u(0) &= u_0 \in Y, \end{cases} \quad (8)$$

where X, Y are Banach spaces, $T_0 \in (0, \infty)$ and let $F : [0, T_0] \times Y \rightarrow X$ is a continous function and u_0 is a given data.

Definition 2.1 (Well posedness) *Let X, Y be Banach spaces, $T_0 \in (0, \infty)$ and let $F : [0, T_0] \times Y \rightarrow X$ be a continous function. We say that the Cauchy problem (8) is **locally well-posed** in Y if:*

1. *There exist $T \in (0, T_0]$ and a function $u \in C([0, T]; Y)$ such that $u(0) = u_0$ and the differential equation is satisfied,*
2. *the problem (8) has at most one solution in $C([0, T]; Y)$,*
3. *the map $u_0 \mapsto u$ is continuous.*

*If the above conditions are satisfied for all $T \in [0, \infty)$, we say that the Cauchy problem is **globally well-posed** in Y .*

Next we present the appropriate spaces for the well posedness results, and we define the kernel considered by Potthas and Graben[16], that significantly extend the others one.

Definition 2.2 *If M is a metric space and $I \subset \mathbb{R}$ a interval then*

- i) $C_b(M) := \{h : M \rightarrow \mathbb{R} \mid h \text{ is continuous and bounded}\}$, *endowed with the norm*

$$\|h\| := \sup_{x \in M} |h(x)|.$$

- ii) $C_b^1(I) := \{g : I \rightarrow \mathbb{R} \mid g \in C^1(I) \text{ and } g' \in C_b(I)\}$, *with the norm*

$$\|g\| := \sup_{t \in I} (|g(t)| + |g'(t)|).$$

- iii) $C_b^{0,1}(\mathbb{R}^m \times [0, \infty)) := \{f : \mathbb{R}^m \times [0, \infty) \rightarrow \mathbb{R} \mid \forall t > 0 \ f(\cdot, t) \in C_b(\mathbb{R}^m), \forall x \in \mathbb{R}^m \ f(x, \cdot) \in C_b^1[0, \infty)\}$, *and norm*

$$\|f\| := \sup_{x \in \mathbb{R}^m, t \geq 0} \left(|f(x, t)| + \left| \frac{df}{dt}(x, t) \right| \right).$$

Definition 2.3 (Synaptic kernel) *The synaptic integral kernel is a function $w : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ such that*

$$w \in L^\infty(\mathbb{R}^m \times \mathbb{R}^m), \quad \|w\|_{L^\infty} = \sup_{x, y \in \mathbb{R}^m} |w(x, y)| \leq C_\infty, \quad (9)$$

and

$$\|w\|_{L_x^\infty L_y^1} := \sup_{x \in \mathbb{R}^m} \|w(x, \cdot)\|_{L_y^1} \leq C_w, \quad (10)$$

that is

$$w \in L_x^\infty L_y^1(\mathbb{R}^m \times \mathbb{R}^m), \quad (11)$$

as a function $x \mapsto w(x, \cdot)$ of $L_x^\infty(\mathbb{R}^m)$ with $w(x, \cdot) \in L_y^1(\mathbb{R}^m)$ for all x , and for some constants $C_\infty, C_w > 0$.

Moreover, w satisfy the Lipschitz condition

$$\|w(x, \cdot) - w(\tilde{x}, \cdot)\|_{L^1} \leq K_w |x - \tilde{x}| \quad x, \tilde{x} \in \mathbb{R}^m \quad (12)$$

for some constant $K_w > 0$.

Definition 2.4 The activation function $f : \mathbb{R} \rightarrow \mathbb{R}$ and the modulation learning function $g : \mathbb{R} \rightarrow \mathbb{R}$ are those such that

$$f, g \in C_b^1(\mathbb{R}), 0 \leq f, g \leq 1. \quad (13)$$

The equation (2) suggest the use of the following operators:

$$(Ju)(x, t) := \int_{\Omega} \omega_{g, \gamma}(x, y) f(u(y, t)) dy, \quad (14)$$

$$(Fu)(x, t) := -u(x, t) + (Ju)(x, t), \quad (15)$$

and

$$(Au)(x, t) := \int_0^t (Fu)(x, s) ds. \quad (16)$$

Note that J is the nonlinear part of equation (2) and we can write the IVP (6)-(7) as:

$$\begin{cases} u' = Fu, \\ u(x, 0) = u_0(x), \end{cases} \quad (17)$$

where $u' \equiv \frac{\partial u}{\partial t}$ and, by integration

$$u = u_0 + Au. \quad (18)$$

It is clear that a solution of (18) is a fixed point of the operator defined by $\varphi \rightarrow u_0 + A\varphi$.

Definition 2.5 A solution of the IVP (17) or (18) is a function $u \in C(\mathbb{R}^m \times [0, \rho))$, for some ρ , such that u satisfies (17) or (18). The flow of the equation $u' = Fu$ is the map

$$(t, u_0) \rightarrow \varphi(t, u_0)$$

defined as

$$\varphi(t, u_0) = u_0 + \int_0^t (F\varphi(\cdot, u_0))(\cdot, s) ds, \quad (19)$$

where

$$\varphi(\cdot, u_0)(x, s) := \varphi(s, u_0)(x). \quad (20)$$

Now, we have the following preliminary estimatives.

Lemma 2.6 *If ω is a synaptic kernel and f, g are activation and modulation learning functions respectively, then the operator J is such that*

$$\|Ju\|_{L_x^\infty L_t^\infty} \leq (1 + \gamma)C_w. \quad (21)$$

Proof 2.7

$$|(Ju)(x, t)| = \left| \int_{\mathbb{R}^m} w(x, y) f(u(y, t)) dy + \gamma \int_{\mathbb{R}^m} w(x, y) g(u(x, t) - u(y, t)) f(u(y, t)) dy \right| \quad (22)$$

$$\leq C_w + \gamma \int_{\mathbb{R}^m} |w(x, y) g(u(x, t) - u(y, t))| dy \quad (23)$$

$$\leq C_w + \gamma \int_{\mathbb{R}^m} |w(x, y)| dy \quad (24)$$

$$\leq (1 + \gamma)C_w. \quad (25)$$

Lemma 2.8 (Global estimates) *Let $u_0 \in C_b(\mathbb{R}^m)$ and u be a solution to (17). Then there is $C > 0$ such that*

$$|u(x, t)| \leq C, \quad \text{for all } (x, t) \in \mathbb{R}^m \times [0, \infty), \quad (26)$$

with $C = \max\{\|u_0\|_\infty, (1 + \gamma)C_w\}$.

Proof 2.9 *By the lemma 2.6 we have that $-c \leq (Ju)(x, t) \leq c$ for some constant c . Then using (6) we obtain the following differential inequalities:*

$$u' \leq -bu + c, \quad (27)$$

$$u' \geq -bu - c, \quad (28)$$

with $b = 1$ and $c = 1 + \gamma C_w$.

If we consider the IVP:

$$\begin{cases} y' = -by + c, \\ y(0) = a, \end{cases} \quad (29)$$

the solution is given by:

$$y(t) = \frac{c}{b} (1 - e^{-bt}) + ae^{-bt}, \quad t > 0. \quad (30)$$

If we plug it in (29), we get

$$y' = b \underbrace{\left(\frac{c}{b} - a\right)}_p e^{-bt}, \quad y(0) = a. \quad (31)$$

Note that if $p < 0$, y will be monotone decreasing and, in that case, $c/b \leq y \leq a$. On the contrary, if $p > 0$, y will be monotone increasing and approaching c/b , then $a \leq y \leq c/b$. Therefore

$$y \leq \max\{|a|, |c/b|\}. \quad (32)$$

Following the same reasoning with

$$\begin{cases} y' = -by - c, \\ y(0) = a, \end{cases} \quad (33)$$

we get

$$y \geq -\max\{|a|, |c/b|\}. \quad (34)$$

Then, changing y by u , a by u_0 , b and c as above, we get the bound which we were looking for:

$$|u(x, t)| \leq \max\{\|u_0\|_\infty, |(1 + \gamma)C_w|\}. \quad (35)$$

Hereafter, the functions ω , f and g are as defined above.

Lemma 2.10 *The operator F is well defined as a map from $C_b^{0,1}(\mathbb{R}^m \times [0, \infty))$ on itself. Moreover, If $u(x, \cdot)$ is continous in t , then $(Ju)(x, \cdot)$ is continous in t too.*

Proof 2.11 *Suppose $u \in C_b^{0,1}(\mathbb{R}^m \times [0, \infty))$. We need to look only at the second term of Fu , $(Ju)(x, t)$. We already saw in the preceding proof that $|(Ju)(x, t)| \leq (1 + \gamma)C_w$.*

Let be $x, \tilde{x} \in \mathbb{R}^m$. We have

$$|(Ju)(x, t) - (Ju)(\tilde{x}, t)| \leq I_1 + I_2 + I_3, \quad (36)$$

where

$$I_1 = \int_{\mathbb{R}^m} |w(x, y) - w(\tilde{x}, y)| |f(u(y, t))| dy \leq k_w |x - \tilde{x}|, \quad (37)$$

$$I_2 = \gamma \int_{\mathbb{R}^m} |[w(x, y) - w(\tilde{x}, y)]g(u(x, t) - u(y, t))| dy \leq \gamma k_w |x - \tilde{x}| \quad (38)$$

and

$$I_3 = \gamma \int_{\mathbb{R}^m} |w(\tilde{x}, y)[g(u(x, t) - u(y, t)) - g(u(\tilde{x}, t) - u(y, t))]| |f(u(y, t))| dy. \quad (39)$$

It is clear that $I_1, I_2 \rightarrow 0$ if $x \rightarrow \tilde{x}$. $I_3 \rightarrow 0$ when $x \rightarrow \tilde{x}$ because of the Lebesgue's Dominated Convergence Theorem. So, we have continuity with respect to x .

Let's see if Fu is continuously differentiable with respect to t . We have this for $u(x, t)$, so we check Ju .

As $f, g \in C_b^1(\mathbb{R})$ and $u(x, \cdot) \in C_b^1([0, \infty))$, we can apply the Leibniz rule and the chain rule to obtain:

$$\begin{aligned} \frac{\partial Ju}{\partial t}(x, t) &= \frac{\partial}{\partial t} \left(\int_{\mathbb{R}^m} w(x, y) [1 + \gamma g(u(x, t) - u(y, t))] f(u(y, t)) dy \right) \\ &= \int_{\mathbb{R}^m} w(x, y) \left\{ \frac{df}{ds}(u(y, t)) \cdot \frac{\partial u}{\partial t}(y, t) \right\} dy + \\ &\quad \gamma \int_{\mathbb{R}^m} w(x, y) \left[g(u(x, t) - u(y, t)) \left\{ \frac{df}{ds}(u(y, t)) \cdot \frac{\partial u}{\partial t}(y, t) \right\} + \right. \\ &\quad \left. f(u(y, t)) \left\{ \frac{dg}{ds}(u(x, t) - u(y, t)) \cdot \left(\frac{\partial u}{\partial t}(x, t) - \frac{\partial u}{\partial t}(y, t) \right) \right\} \right] dy. \end{aligned} \quad (40)$$

Because df/ds , du/ds and dg/ds are continuous and bounded, we get that $\frac{\partial Ju}{\partial t}(x, t)$ is continuous and $\sup_{x, t} |\frac{\partial Ju}{\partial t}(x, t)| < \infty$.

Finally, if $u(x, \cdot)$ is continuous in t then $(Ju)(x, \cdot)$ is continuous in t by application of the dominated convergence theorem as in the previous steps.

Lemma 2.12 u is a solution of (17) if and only if it is a solution of (18).

Proof 2.13 If u is a solution of (18), we need to ensure that it is differentiable with respect to time. However, Fu is continuous, so $\int_0^t (Fu)(x, s) ds$ is differentiable with respect to t . Now we can differentiate (18) and u satisfies (17).

If u is a solution of (17), we can integrate it to satisfy (18).

Remark 2.14 The above lemmas are another versions of those obtained by Potthas and Grabem in the section 2 [...] in the case $\gamma = 0$ (when the Amary's model works). The proofs follow those same ideas.

3 Well posedness in $C_b(\mathbb{R}^n)$

For $\rho > 0$ given, we write $X_\rho := C_b(\mathbb{R}^n \times [0, \rho]) = C([0, \rho]; C_b(\mathbb{R}^n))$.

Lemma 3.1 The operator A is well defined as an map on X_ρ .

Proof 3.2 Let $u \in X_\rho$. That $(Au)(\cdot, t) \in C_b(\mathbb{R}^m)$ is a direct consequence of previous lemmas. We only need to see that $(Au)(x, \cdot)$ is continuous. Let us take $t_1, t_2 \in (0, \rho]$. Then for x fixed,

$$|(Au)(x, t_1) - (Au)(x, t_2)| = \left| \int_0^{t_1} (Fu)(x, s) ds - \int_0^{t_2} (Fu)(x, s) ds \right|, \quad (42)$$

$$= \left| \int_{t_2}^{t_1} (-u(x, s) + (Ju)(x, s)) ds \right|, \quad (43)$$

$$\leq \int_{t_2}^{t_1} |u(x, s)| ds + \int_{t_2}^{t_1} |(Ju)(x, s)| ds, \quad (44)$$

$$\leq C|t_1 - t_2| + (1 + \gamma)C_w|t_1 - t_2|, \quad (45)$$

so, $(Au)(x, \cdot)$ is continuous on $(0, \rho]$. The continuity at zero can be seen by noting that

$$|(Au)(x, t_1) - (Au)(x, 0)| = \left| \int_0^{t_1} (-u(x, s) + (Ju)(x, s)) ds \right|. \quad (46)$$

$$\leq Ct_1 + (1 + \gamma)C_w(t_1). \quad (47)$$

This completes the proof.

Theorem 3.3 The operator A is a contraction in X_ρ for $\gamma = \gamma_0$ fixed and $\rho > 0$ sufficiently small.

Proof 3.4 The operator A can be divided in three different operators as

$$A = A_1 + A_2 + A_3, \quad (48)$$

where A_1 is the linear integral operator

$$(A_1 v)(x, t) := - \int_0^t v(x, s) ds, \quad 0 \leq t \leq \rho, \quad (49)$$

and A_2 and A_3 , are the nonlinear integral operators

$$(A_2 v)(x, t) := \int_0^t \int_{\mathbb{R}^m} w(x, y) f(v(y, s)) dy ds, \quad 0 \leq t \leq \rho, \quad (50)$$

$$(A_3 v)(x, t) := \gamma \int_0^t \int_{\mathbb{R}^m} w(x, y) g(v(x, s) - v(y, s)) f(v(y, s)) dy ds, \quad 0 \leq t \leq \rho. \quad (51)$$

Let $u := u_1 - u_2$. Then A_1 satisfies

$$\|A_1 u\|_\rho \leq \rho \|u\|_\rho. \quad (52)$$

Since $f \in C_b^1(\mathbb{R})$, then there is a constant L such that $|f(s) - f(\tilde{s})| \leq L|s - \tilde{s}|$ for all $s, \tilde{s} \in \mathbb{R}$. Therefore

$$\int_{\mathbb{R}^m} |w(x, y)| |f(u_1(y, t)) - f(u_2(y, t))| dy \leq LC_w \|u_1 - u_2\|_\rho, \quad (53)$$

and

$$|(A_2 u_1)(x, t) - (A_2 u_2)(x, t)| \leq \int_0^t \int_{\mathbb{R}^m} |w(x, y)| |f(u_1(y, s)) - f(u_2(y, s))| dy ds \quad (54)$$

$$\leq \rho L C_w \|u_1 - u_2\|_\rho, \quad (55)$$

so

$$\|A_2 u_1 - A_2 u_2\|_\rho \leq \rho L C_w \|u_1 - u_2\|_\rho. \quad (56)$$

Finally, we obtain an estimative for A_3 .

Considering that function g is also Lipschitz with Lipschitz constant K , we have

$$\left| \int_{\mathbb{R}^m} w(x, y) [g(u_1(x, t) - u_1(y, t)) f(u_1(y, t)) - g(u_2(x, t) - u_2(y, t)) f(u_2(y, t))] dy \right| \quad (57)$$

$$\begin{aligned} &\leq \int_{\mathbb{R}^m} |w(x, y)| |g(u_1(x, t) - u_1(y, t))| |f(u_1(y, t)) - f(u_2(y, t))| dy + \\ &\quad \int_{\mathbb{R}^m} |w(x, y)| |f(u_2(y, t))| |g(u_1(x, t) - u_1(y, t)) - g(u_2(x, t) - u_2(y, t))| dy \end{aligned} \quad (58)$$

$$\begin{aligned} &\leq L \int_{\mathbb{R}^m} |w(x, y)| |u_1(y, t) - u_2(y, t)| dy + \\ &\quad K \int_{\mathbb{R}^m} |w(x, y)| |u_1(x, t) - u_2(x, t) + u_2(y, t) - u_1(y, t)| dy \end{aligned} \quad (59)$$

$$\leq L \|u_1 - u_2\|_\infty \int_{\mathbb{R}^m} |w(x, y)| dy + 2K \|u_1 - u_2\|_\infty \int_{\mathbb{R}^m} |w(x, y)| dy \quad (60)$$

$$\leq C_w (L + 2K) \|u_1 - u_2\|_\rho, \quad (61)$$

then,

$$\begin{aligned} |(A_3 u_1 - A_3 u_2)(x, t)| &\leq \gamma \int_0^t \int_{\mathbb{R}^m} |w(x, y) [g(u_1(x, s) - u_1(y, s)) f(u_1(y, s)) - \\ &\quad g(u_2(x, s) - u_2(y, s)) f(u_2(y, s))]| dy ds \end{aligned} \quad (62)$$

$$\leq \gamma \rho C_w (L + 2K) \|u_1 - u_2\|_\rho, \quad (63)$$

and therefore,

$$\|A_3 u_1 - A_3 u_2\|_\rho \leq \gamma \rho (L + 2K) C_w \|u_1 - u_2\|_\rho. \quad (64)$$

Summarizing, we have

$$\|A u_1 - A u_2\|_\rho \leq \rho [1 + L C_w + \gamma (L + 2K) C_w] \|u_1 - u_2\|_\rho, \quad (65)$$

thus, $A = A_1 + A_2 + A_3$ is a contraction for $0 < q := \rho [1 + L C_w + \gamma (L + 2K) C_w] < 1$.

Theorem 3.5 (Local existence of solutions) *Let L and K being the Lipschitz constants of f and g respectively, $\gamma = \gamma_0$ fixed and $\rho > 0$ such that $0 < q < 1$. Then for all $u_0 \in C_b(\mathbb{R}^m)$ there exists a unique solution $u \in X_\rho$ for the IVP (17).*

Proof 3.6 *The space X_ρ is a Banach space and the operator $\tilde{A}u = u_0 + Au$ is a contraction on X_ρ . Then the equation $u = \tilde{A}u$ has an unique fixed point u^* , which corresponds with the solution to (18) and, by lemma 2.12, to the solution of (17). Because the operator \tilde{A} is defined on all the space X_ρ , the solution to (17) is unique.*

Theorem 3.7 (Global existence of solutions) *For all $u_0 \in C_b(\mathbb{R}^m)$ there is a solution $u \in C_b(\mathbb{R}^m \times [0, \infty))$.*

Proof 3.8 *Note that neither (17) nor the obtained estimates depend on time explicitly and ρ in theorem 3.5 does not depend on the initial data u_0 . Also, the theorem 2.8 guarantees that the solution is bounded. Then, we can repeat the presented argument for initial data $u(x, \rho) \in C_b(\mathbb{R}^m)$ and extend the solution to an interval $[\rho, 2\rho]$. Iterating this, we can cover the entire interval $[0, \infty)$.*

Theorem 3.9 (Continuous dependence on initial data) *Let u and v be solutions to (18), with initial data u_0 and v_0 respectively. Then, for a fixed γ and for every $\rho > 0$, there exists a constant $C = C(\rho)$ such that*

$$\|u - v\|_\rho \leq C \|u_0 - v_0\|_{C_b(\mathbb{R}^m)}. \quad (66)$$

Proof 3.10 *From theorem 3.3 we have*

$$\|Au - Av\|_\rho \leq q \|u - v\|_\rho, \quad (67)$$

choosing $\rho > 0$ such that $q := \rho[1 + LC_w + \gamma(L + 2K)C_w] < 1$.

Then if u and v are solutions to (18), with initial data u_0 and v_0 respectively, we obtain

$$\|u - v\|_\rho \leq \|u_0 - v_0\|_{C_b(\mathbb{R}^m)} + \|Au - Av\|_\rho \leq \|u_0 - v_0\|_{C_b(\mathbb{R}^m)} + q \|u - v\|_\rho, \quad (68)$$

so

$$\|u - v\|_\rho \leq C \|u_0 - v_0\|_{C_b(\mathbb{R}^m)} \quad (69)$$

with $C = \frac{1}{1-q}$.

Chosen that ρ , we can repeat the latter argument in $[\rho, 2\rho]$ and we will obtain the same estimative. Repeating the argument many times as necessary, cover any interval.

The following result is an immediate consequence of theorem 2.8, and says that a purely excitatory network, remains excitatory

Theorem 3.11 (Positivity) Let $u_0 \in C_b(\mathbb{R}^m)$, and $\gamma > 0$. Suppose $u_0(x) \geq 0$ for all $x \in \mathbb{R}^m$. If the synaptic kernel $w : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$, satisfy:

$$w(x, y) > 0 \quad \forall x, y \in \mathbb{R}^m, \quad (70)$$

then

$$u(x, t) \geq 0 \quad \forall x \in \mathbb{R}^m, t \geq 0. \quad (71)$$

Proof 3.12 We already saw in theorem 2.8 that $c \leq u \leq a$ or $a \leq u \leq c$, with $c > 0$ and $a = u_0$. Therefore, we can conclude that $u(x, t) \geq 0$ for all $t \in [0, \infty)$.

Now, we will see that the Amari's model is a good approximation of the model proposed by Abbassian et al. More precisely:

Theorem 3.13 (Annulling Plasticity Limit) Let u^γ and u the solutions to the problems (6)-(7) and (1) with initial data u_0^γ and u_0 respectively. If

$$\lim_{\gamma \rightarrow 0} \|u_0^\gamma - u_0\|_{C_b(\mathbb{R}^m)} = 0, \quad (72)$$

then:

$$\lim_{\gamma \rightarrow 0} \|u^\gamma - u\|_{C_b(\mathbb{R}^m \times [0, \infty))} = 0. \quad (73)$$

Proof 3.14 We can put the IVP (1) like a Volterra integro differential equation too, similar to (18), but with ω instead of $\omega_{g, \gamma}$. Then we have the contraction operators A and A^γ , corresponding with the Amari's and Abbassian's models, respectively,

Let $\rho > 0$ be fixed and $0 \leq t < \rho$. We have the following estimatives

$$N_1 := \int_{\mathbb{R}^m} |w(x, y)| |f(u(y, t)) - f(u^\gamma(y, t))| dy \leq LC_w \|u - u^\gamma\|_{C_b(\mathbb{R}^m \times [0, \infty))} \quad (74)$$

$$N_2 := \int_{\mathbb{R}^m} |w(x, y)| |g(u^\gamma(x, t) - u^\gamma(y, t))| |f(u^\gamma(y, t))| dy \leq KC_w \|u - u^\gamma\|_{C_b(\mathbb{R}^m \times [0, \infty))}. \quad (75)$$

Then

$$|Au(x, t) - A^\gamma u^\gamma(x, t)| \leq \int_0^t (|u(x, s) - u^\gamma(x, s)| + N_1 + \gamma N_2) ds \quad (76)$$

$$\leq \rho[1 + (L + \gamma K)C_w] \|u - u^\gamma\|_{C_b(\mathbb{R}^m \times [0, \infty))}, \quad (77)$$

so,

$$|u(x, t) - u^\gamma(x, t)| \leq |u_0(x) - u_0^\gamma(x)| + |Au(x, t) - A^\gamma u^\gamma(x, t)| \quad (78)$$

$$\leq \|u_0 - u_0^\gamma\|_{C_b(\mathbb{R}^m)} + \rho[1 + (L + \gamma K)C_w] \|u - u^\gamma\|_{C_b(\mathbb{R}^m \times [0, \infty))}. \quad (79)$$

Therefore, if ρ is chosen such that $\rho[1 + (L + \gamma K)C_w] < 1$, we have

$$\|u - u^\gamma\|_\rho \leq \frac{1}{1 - \rho[1 + (L + \gamma K)C_w]} \|u_0 - u_0^\gamma\|_{C_b(\mathbb{R}^m)}. \quad (80)$$

Then $u^\gamma \rightarrow u$ uniformly on $\mathbb{R}^n \times [0, \rho]$ when $\gamma \rightarrow 0$. Because the estimatives do not depend explicitly of time, we can iterate this procedure in $[\rho, 2\rho)$ and so on, with the same ρ , to cover the entire interval $[0, \infty)$.

4 Well posedness in $L^1(\Omega)$

The brain is not a continuous medium and it receives and emits no continuous signals, then is interesting to look for solutions for non continuous data in a bounded spatial domain.

So, in this section we are going to consider the model of Abbassian et al

$$\begin{cases} u_t(x, t) = -u(x, t) + \int_{\Omega} \omega_{g, \gamma} f(u(y, t)) dy, & (x, t) \in \Omega \times (0, \infty) \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (81)$$

with $u_0 \in L^1(\Omega)$ and $\Omega \subset \mathbb{R}^m$ bounded.

Let $\rho > 0$ fixed. We consider the space of continuous functions with respect to t , defined on $[0, \rho]$ with values in $L^1(\Omega)$ and we denote that space by

$$X_\rho := C([0, \rho]; L^1(\Omega)). \quad (82)$$

Now, a solution of (81) will be a function $u \in X_\rho$ such that it satisfies (81).

The following results ensures the well posedness of the Cauchy problem (81).

Lemma 4.1 *If u is solution of the IVP (81), then it is bounded.*

Proof 4.2 *For a solution u of (81) we have that*

$$u = e^{-t}u_0 + \int_0^t e^{s-t}(Ju)(x, s)ds, \quad (83)$$

then

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq e^{-t}\|u_0(x)\|_{L^1(\Omega)} + \int_0^t e^{s-t} \|Ju\|_{L^1(\Omega)} ds, \quad (84)$$

$$\leq e^{-t}\|u_0(x)\|_{L^1(\Omega)} + C_w|\Omega|(1 - e^{-t}), \quad (85)$$

$$\leq \|u_0(x)\|_{L^1(\Omega)} + C_w|\Omega|, \quad (86)$$

so,

$$\sup_t \|u(\cdot, t)\|_{L^1(\Omega)} \leq \infty. \quad (87)$$

Theorem 4.3 *Let $u_0 \in L^1(\Omega)$. The operator $A : X_\rho \rightarrow X_\rho$, as defined in (16), is well defined.*

Proof 4.4 *Let $0 \leq t < \rho$, $u \in X_\rho$. Then, we have the following estimate for the operator J :*

$$\|(Ju)(\cdot, t)\|_{L^1(\Omega)} = \int_{\Omega} \left| \int_{\Omega} w(x, y) [1 + \gamma g(u(x, t) - u(y, t))] f(u(y, t)) dy \right| dx \quad (88)$$

$$\leq \int_{\Omega} \int_{\Omega} |w(x, y) [1 + \gamma g(u(x, t) - u(y, t))] f(u(y, t))| dy dx \quad (89)$$

$$\leq \int_{\Omega} \int_{\Omega} |w(x, y)| dy dx \quad (90)$$

$$\leq C_w |\Omega|, \quad (91)$$

where $|\Omega|$ is the measure of Ω .

Since $u(\cdot, t) \in L^1(\Omega)$ we have $(Fu)(\cdot, t) \in L^1(\Omega)$, so $(Au)(\cdot, t) \in L^1(\Omega)$ by the application of Fubini's theorem. The $(Au)(x, \cdot)$ can be shown to be continuous on $[0, \rho]$ in an analogous manner as was done in the lemma 3.1.

Theorem 4.5 *Let $u, v \in X_\rho$ are solutions to (81). Then there exists a constant $C = C(\Omega, w, g, f, \gamma)$ such that*

$$\|Au - Av\|_\rho \leq \rho C \|u - v\|_\rho \quad (92)$$

Proof 4.6 *Let us divide the operator A in three operators A_1 , A_2 and A_3 , as previously (eqs (49), (50), (51)), such that: $A = A_1 + A_2 + A_3$.*

We have the following estimate for the linear operator A_1

$$\|(A_1 u)(\cdot, t) - (A_1 v)(\cdot, t)\|_{L^1(\Omega)} \leq \rho \|u - v\|_\rho. \quad (93)$$

For A_2 , we obtain the following estimate

$$\|(A_2 u)(\cdot, t) - (A_2 v)(\cdot, t)\|_{L^1(\Omega)} \leq \int_0^t \int_{\Omega} \int_{\Omega} |w(x, y)| |f(u(y, s)) - f(v(y, s))| dy dx ds, \quad (94)$$

$$\leq L \int_0^t \int_{\Omega} |u(y, s) - v(y, s)| \int_{\Omega} |w(x, y)| dx dy ds, \quad (95)$$

$$\leq \rho L C_w |\Omega| \|u - v\|_\rho. \quad (96)$$

Finally, we can obtain an estimate for A_3 :

$$\begin{aligned}
\|(A_3 u)(\cdot, t) - (A_3 v)(\cdot, t)\|_{L^1(\Omega)} &\leq \gamma \int_0^t \int_\Omega \int_\Omega |w(x, y)| |g(u(x, s) - u(y, s))f(u(y, s)) - \\
&\quad g(v(x, s) - v(y, s))f(v(y, s))| dy ds dx \quad (97) \\
&= \gamma \int_0^t \int_\Omega \int_\Omega |w(x, y)| |g(u(x, s) - u(y, s))f(u(y, s)) - \\
&\quad g(v(x, s) - v(y, s))f(v(y, s))| dy dx ds \quad (98) \\
&= \gamma \int_0^t \int_\Omega \int_\Omega |w(x, y)| |g(u(x, s) - u(y, s))f(u(y, s)) - \\
&\quad g(v(x, s) - v(y, s))f(u(y, s)) + g(v(x, s) - v(y, s))f(u(y, s)) - \\
&\quad g(v(x, s) - v(y, s))f(v(y, s))| dy dx ds \quad (99) \\
&\leq \gamma \left(\int_0^t \int_\Omega \int_\Omega |w(x, y)| |g(u(x, s) - u(y, s)) - \\
&\quad g(v(x, s) - v(y, s))| dy dx ds + \right. \\
&\quad \left. \int_0^t \int_\Omega \int_\Omega |w(x, y)| |f(u(y, s)) - f(v(y, s))| dy dx ds \right) \quad (100) \\
&\leq \gamma \left(K \int_0^t \int_\Omega \int_\Omega |w(x, y)| |u(x, t) - v(x, t)| dy dx ds + \right. \\
&\quad \left. (L + K) \int_0^t \int_\Omega \int_\Omega |w(x, y)| |u(y, t) - v(y, t)| dy dx ds \right) \quad (101) \\
&\leq \rho \gamma K C_w |\Omega| \|u - v\|_\rho + \rho \gamma (L + K) C_w |\Omega| \|u - v\|_\rho \quad (102) \\
&= \rho \gamma (L + 2K) C_w |\Omega| \|u - v\|_\rho. \quad (103)
\end{aligned}$$

Therefore

$$\|Au - Av\|_\rho \leq \rho \|u - v\|_\rho + \rho L C_w |\Omega| \|u - v\|_\rho + \rho \gamma (L + 2K) C_w |\Omega| \|u - v\|_\rho \quad (104)$$

$$= \rho [1 + C_w |\Omega| (L + \gamma (L + 2K))] \|u - v\|_\rho \quad (105)$$

$$= \rho C \|u - v\|_\rho, \quad (106)$$

with $C := 1 + C_w |\Omega| (L + \gamma (L + 2K))$ independent of the time.

Theorem 4.7 (Existence and uniqueness in $L^1(\Omega)$) For all $u_0 \in L^1(\Omega)$, there exists a unique solution $u \in C[[0, \infty); L^1(\Omega)]$ for the IVP (81).

Proof 4.8 Choose ρ as in theorem 4.5 such that

$$C := \rho [1 + C_w |\Omega| (L + \gamma (L + 2K))] < 1. \quad (107)$$

Then the operator A is a contraction in the Banach space X_ρ . The existence and uniqueness follows from the Banach Fixed Point Theorem applied to the operator $\tilde{A} : X_\rho \rightarrow X_\rho$ given by $\tilde{A} = u_0 + A$, i.e.

$$(\tilde{A}u)(x, t) = u_0(x) + (Au)(x, t), \quad (108)$$

which is also a contraction due to A . Now, this local result can be extended to the whole interval $[0, \infty)$ noting that the estimatives just obtained are the same and independent of initial data. For initial data $u(x, \rho) \in L^1(\Omega)$, a similar result can be obtained for $[\rho, 2\rho]$, and so on.

Theorem 4.9 (continuous dependence on initial data) *Let u and v be solutions to 81, with initial data u_0 and v_0 respectively. Then, for a fixed γ , there exists $\rho > 0$ and a constant $C = C(\rho)$ such that*

$$\|u - v\|_\rho \leq C \|u_0 - v_0\|_{L^1(\Omega)} \quad (109)$$

Proof 4.10 *This is consequence of the the operator being a contraction, similar as in theorem 3.9.*

5 Stationary solutions

The following results guaranties the existence of stationary solutions in a general way, improving results obtained in [6]

Lemma 5.1 *Suppose that $u^\gamma = u^\gamma(x, t)$ is a global solution of (6) in $X_\rho = C([0, +\infty); C_b(\Omega))$. Then the family $\{u^\gamma(\cdot, t)\}_{t \geq 0}$ is equicontinuous in $C_b(\Omega)$ if γ is small enough and Ω is compact. In this case, we can choose a sequence of times $t_1 < t_2 < \dots, t_n \rightarrow +\infty$; such that $\{u^\gamma(\cdot, t_n)\}_{n \in \mathbb{N}}$ converges uniformly to a function $u_\infty = u_\infty(x)$ in $C(\Omega)$.*

Proof 5.2 *A global solution of (6) is*

$$u(x, t) = e^{-t}u_0(x) + \int_0^t e^{s-t}(Ju)(x, s)ds, \quad (110)$$

so

$$|u(x, t) - u(\tilde{x}, t)| \leq |u_0(x) - u_0(\tilde{x})| + \int_0^t e^{s-t} |(Ju)(x, s) - (Ju)(\tilde{x}, s)| ds, \quad (111)$$

$$\leq |u_0(x) - u_0(\tilde{x})| + \|(Ju)(x, \cdot) - (Ju)(\tilde{x}, \cdot)\|_\infty. \quad (112)$$

But

$$|(Ju)(x, t) - (Ju)(\tilde{x}, t)| \leq I_1 + I_2 + I_3, \quad (113)$$

whit

$$I_1 \leq k_w |x - \tilde{x}|, \quad (114)$$

$$I_2 \leq \gamma k_w |x - \tilde{x}| \quad (115)$$

and

$$I_3 = \gamma \int_{\mathbb{R}^m} |w(\tilde{x}, y)[g(u(x, t) - u(y, t)) - g(u(\tilde{x}, t) - u(y, t))]| |f(u(y, t))| dy. \quad (116)$$

As $g(\delta) = e^{-\delta^2}$ es a contraction, then

$$I_3 \leq \gamma C_w |u(x, t) - u(\tilde{x}, t)|, \quad (117)$$

therefore

$$\|u(x, \cdot) - u(\tilde{x}, \cdot)\|_\infty \leq |u_0(x) - u_0(\tilde{x})| + k_w |x - \tilde{x}| + \gamma k_w |x - \tilde{x}| + \gamma C_w \|u(x, \cdot) - u(\tilde{x}, \cdot)\|_\infty. \quad (118)$$

If Ω is compact then u_0 is uniformly continuous, so we can choose γ small enough such that $\{u^\gamma(\cdot, t)\}_{t \geq 0}$ is equicontinuous. Since this family is bounded, the Arzela theorem guarantees a sequence $\{u^\gamma(\cdot, t_n)\}_{n \in \mathbb{N}}$ convergent uniformly to a function $u_\infty = u_\infty(x)$ in $C(\Omega)$.

Lemma 5.3 Suppose that $\varphi_n \rightarrow \varphi$ uniformly in S . Let x a limit point of S and suppose that $\lim_{t \rightarrow x} \varphi_n(t) = a_n$. Then $\{a_n\}_n$ is convergent and $\lim_{n \rightarrow \infty} a_n = \lim_{t \rightarrow x} \varphi(t)$, that is

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow x} \varphi_n(t) = \lim_{t \rightarrow x} \lim_{n \rightarrow \infty} \varphi_n(t). \quad (119)$$

Lemma 5.4 Suppose that $\varphi_n \rightarrow \varphi$ uniformly in S and $\{\varphi_n\}_{n \in \mathbb{N}}$ is uniformly bounded, that is, there is a $M > 0$ such that $|\varphi_n(x)| \leq M$ for all x in S , $n \in \mathbb{N}$. If ψ is a continuous function on the closed the ball $\bar{B}(0; M)$, then $(\psi \circ \varphi_n) \rightarrow (\psi \circ \varphi)$ uniformly in S .

The above lemmas are well known results from advanced calculus.

Finally, we have

Theorem 5.5 If Ω is compact, then for sufficiently small values of $\gamma \geq 0$ the equation (6) has a stationary solution in Ω .

Proof 5.6 Let $\{u^\gamma(\cdot, t_n)\}_{n \in \mathbb{N}}$ with limit u_∞ , as in lemma 5.1. We have that $f(u^\gamma(\cdot, t_n)) \rightarrow f(u_\infty(\cdot))$ uniformly in Ω because the previous lemma. Then

$$\int_{\Omega} \omega_{g, \gamma}(\cdot, y) f(u^\gamma(y, t_n)) dy \rightarrow \int_{\Omega} \omega_{g, \gamma}(\cdot, y) f(u_\infty(y)) dy \quad (120)$$

uniformly in Ω because of $\int_{\Omega} |\omega(x, y)| dy \leq C_\infty |\Omega|$.

As

$$\partial_t u^\gamma(\cdot, t) = \lim_{h \rightarrow 0} \frac{u^\gamma(\cdot, t+h) - u^\gamma(\cdot, t)}{h} \quad (121)$$

exists, and

$$\lim_{n \rightarrow \infty} \delta_{t_n}(h) = 0 \quad (\text{uniformly}) \quad (122)$$

with

$$\delta_{t_n}(h) := \frac{u^\gamma(\cdot, t_n + h) - u^\gamma(\cdot, t_n)}{h}, \quad (123)$$

then we can take limit in equation (6) and get

$$u_\infty(x) = \int_{\Omega} \omega_{g,\gamma}(x, y) f(u_\infty(y)) dy, \quad \forall x \in \Omega. \quad (124)$$

Therefore, u_∞ is a stationary solution of the equation (6).

5.1 The result of learning: the gain field equation

Now we develop a novel connection between learning kernels and gain field equations. In the stationary case, the kernel $\omega_{g,\gamma}$ reaches its final form

$$\bar{\omega}_{g,\gamma}(x, y) = \omega(x, y) G(u_\infty(x), u_\infty(y)) = \omega(x, y) [1 - g(u_\infty(x) - u_\infty(y))]. \quad (125)$$

This kernel represents the distribution of synaptic weights *after* the learning takes place. We can now plug this kernel $\bar{\omega}_{g,\gamma}$ into a new equation

$$u_t(x, t) + u(x, t) = \int_{\Omega} \omega(x, y) G(u_\infty(x), u_\infty(y)) f(u(y, t)) dy, \quad (126)$$

and study the dynamics of the Amari neural field under the learned spatial patterns. Note that in this equation, the kernel does not change anymore in an activity dependent manner. The domain Ω can be bounded or unbounded. The learned kernel can be divided in *pre-synaptic* and *post-synaptic* contributions by using Mercer's theorem [21, 22], which states that positive semidefinite kernels on Ω can be decomposed as

$$G(x, y) = \sum_i \sigma_i \phi_i(x) \phi_i(y) \quad (127)$$

where ϕ_i and σ_i are sequences of orthonormal eigenfunctions and the corresponding eigenvalues, $i = 1, \dots, \infty$. This decomposition can be nicely interpreted as the presynaptic and postsynaptic effects of learning. It is known that, presynaptically, learning increases the probability of the neurotransmitter release, while postsynaptically it affects the density of receptors[5]. Presynaptic effects can be translated in terms of an effective gain field. For that we have then the following estimate

$$\begin{aligned} \sum_i \sigma_i \phi_i(x) \phi_i(y) &\leq \left(\sum_i |\sigma_i \phi_i(x) \phi_i(y)| \right)^2 \\ &\leq \sum_i \sigma_i |\phi_i(x)|^2 \sum_i \sigma_i |\phi_i(y)|^2 \\ &= \left[\max_{x \in \Omega} G(x, x) \right] \sum_i \sigma_i |\phi_i(y)|^2. \end{aligned} \quad (128)$$

The last term in the inequality

$$\phi_{\text{pre}}(y) = K_{\text{pre}} \sum_i \sigma_i |\phi_i(y)|^2, \quad (129)$$

is called *pre-synaptic gain field*, for some constant $K_{\text{pre}} > 0$. The pre-synaptic gain field equation is then defined as

$$u_t(x, t) + u(x, t) = \int_{\Omega} \omega(x, y) \phi_{\text{pre}}(y) f(u(y, t)) dy. \quad (130)$$

It is easy to show that the particular conditions on the learning kernel (definition 2.4) imply that it is positive semidefinite, that is $\int \int \phi(x) G(x, y) \phi(y) dx dy \geq 0$ for all $\phi \in L^2(\Omega)$. We state the following lemma without proof

Lemma 5.7 *Let u^γ and u^p be solutions of (2) and (130), then, $u^p(x, t) > u^\gamma(x, t)$ for all $t > T$ for some $T > 0$.*

We now concentrate in a particular form of (129) and synaptic kernels that allows us to introduce a new methodology to analyze gain fields emerging from learning processes. For $K_{\text{pre}} = 1/\lambda$,

$$\phi_{\text{pre}}(x) = \frac{1}{\lambda} (k^2 - V(x)), \quad (131)$$

This function represents a spatial gain distribution relative to a background or base gain k^2 . The synaptic kernel takes the form of an exponentially decaying function and it is assumed spatially homogeneous, positive (effectively excitatory) and isotropic

$$w(x, y) = w(|x - y|) = \frac{1}{2} e^{-\lambda|x-y|}, \quad \lambda > 0. \quad (132)$$

For linear firing rate mappings $f(u(x)) = u(x)$ and under the previous assumptions, equation (1) becomes

$$u_t(x, t) = -u(x, t) + \frac{1}{2\lambda} \int_{\mathbb{R}^m} e^{-\lambda|x-y|} \phi_{\text{pre}}(y) u(y, t) dy, \quad (133)$$

This model has been studied before in the literature for the case that we call the *free network* $V = 0$, applied for problems of non-local diffusion and in population dynamics. This model can also be related to the time independent Schrödinger equation, a relation which allows us to study presynaptic gain fields in analogy to *quantum wells*. First, suppose that the solutions to equation (133) approach an stationary state $u(x)$, given by

$$u(x) = \frac{1}{2\lambda} \int_{\mathbb{R}} e^{-\lambda|x-y|} P(y) u(y) dy, \quad (134)$$

which can be written as

$$u(x) = w * [f \circ u(\cdot, t)](x), \quad (135)$$

where $*$ denotes spatial convolution and \circ the function composition. Then, we can apply the Fourier Transform (FT) with respect to the spatial variable to obtain

$$\hat{u} = \widehat{w * f \circ u}, \quad (136)$$

and using the fact that $\widehat{w * f} = \hat{w} \hat{f}$ we get

$$\hat{u}(\xi) = \frac{1}{\xi^2 + \lambda^2} \widehat{f \circ u}(\xi).$$

Multiplying by \hat{w} on both sides and taking the inverse FT and using (131) we can write

$$\lambda^2 u(x) - u_{xx}(x) = [k^2 - V(X)]u(x), \quad (137)$$

that is

$$-\frac{d^2 u}{dx^2}(x) + V(x)u(x) = Eu(x), \quad (138)$$

with $E = k^2 - \lambda^2$, which is the time-independent Schrödinger equation.

We believe this relationship allow us to frame three questions as inquiries about the eigenfunction $u(x)$:

1. Under which conditions can a focus of asynchronous activity be propagated from a region of high gain to a region of low gain?
2. Under which conditions of the gain field can regions of high (up states) and low (down states) coexist in the same spatial domain?
3. Under which conditions can this network support bumps?

We answer those questions in [23]. In particular, we show that stable solutions are possible in the presence of gain fields even if the neural field has completely excitatory synaptic kernels and unbounded firing rate mappings.

Remark

Note that if V is a square well potential with height k^2 , then, the integral in (134) is over a compact interval, therefore, the existence of u is well justified.

6 Conclusions

We have shown that the model proposed by Abbassian et al. is well defined for general choices of the function f , w and g . Moreover, our results guarantee the existence of solutions in general functional spaces and the existence of the stationary state, which put the conclusions reached in that paper in a solid

analytic ground for further development.

In the case in which f is a heaviside function, the IVP turns out to be ill posed [Cordero y Pinilla (to be published)]. This is because the equation's flow is discontinuous and, therefore, a fixed point argument can not be used in X_ρ . In that case, one expects to proof the existence of weaker solutions by using compactness arguments as in [16]. Other aspects like the speed, and decay of traveling waves can also be studied in this, more general, case.

The final form of the kernel is important in that it represent the final structure of the synaptic weight after learning. We have shown that this structure can be decomposed in pre-synaptic and post-synaptic distributions, of which we have focused in the pre-synaptic ones. We derive a form of the time independent Schrödinger equation that can be used to study particular distributions of gains. This methodology is further developed in future work.

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